

Section 1.2. Initial-Value Problems

We are interested in problems in which we seek a solution $y(x)$ of a diff. eq. so that $y(x)$ also satisfies certain prescribed side conditions, that is, conditions that are imposed on the unknown function $y(x)$ and its derivatives at a number x_0 .

The general form of an n th-order diff. eq. subject to n side conditions specified at x_0 :

$$\text{Solve: } \frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \quad (1)$$

$$\text{Subject to: } y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}.$$

where $y_0, y_1, \dots, y_{n-1}$ are arbitrary constant. This is called an **n th-order initial-value problem (IVP)**.

The values of y and its first $n-1$ derivatives at x_0 , $y_0, y_1, \dots, y_{n-1}$, are called **initial condition (IC)**.

GEOMETRIC INTERPRETATION

The case $n=1$ and $n=2$ in (1),

$$\text{Solve: } \frac{dy}{dx} = f(x, y) \quad (2)$$

$$\text{Subject to: } y(x_0) = y_0$$

and

$$\begin{aligned} \text{Solve: } & \frac{d^2 y}{dx^2} = f(x, y, y') \\ \text{Subject to: } & y(x_0) = y_0, y'(x_0) = y_1. \end{aligned} \quad (3)$$

are examples of first- and second-order initial-value problems.

For (2) we are finding a solution of the eq $y' = f(x, y)$ on an interval I containing x_0 so that its graph passes through the specific point (x_0, y_0) .

For (3) we would like to find a solution $y(x)$ of $y'' = f(x, y, y')$ s.t. its graph not only passes through (x_0, y_0) but the slope of the curve at this point is also the number y_1 .

Example 1.

(a) $y = ce^x$ is a one-parameter family of solutions of the equation $y' = y$. If we impose an initial condition $y(0) = 4$, then substituting $x = 0, y = 4$ in the family determines the constant $4 = ce^0 = c$. Thus $y = 4e^x$ is a solution of the IVP $y' = y, y(0) = 4$.

(b) Now if we demand that a solution curve pass through the point $(2, 5)$ rather than $(0, 4)$, then $y(2) = 5$ yields $5 = ce^2$, or $c = 5e^{-2}$. Thus $y = 5e^{x-2}$ is a solution of the IVP:

$$y' = y, \quad y(2) = 5.$$

□

The interval I of definition of the solution depends on the initial condition.

Example 2. The first-order diff equation
$$y' + 2xy^2 = 0$$
has a solution $y = \frac{1}{x^2 + c}$.

If we impose the initial condition $y(0) = -1$, then $c = -1$. Thus $y = \frac{1}{x^2 - 1}$.

The function $y = \frac{1}{x^2 - 1}$ has domain $\mathbb{R} - \{\pm 1\}$. So the interval I of definition of the solution $y = \frac{1}{x^2 - 1}$ is an interval containing 0 in which $y(x)$ is defined and differentiable.

$\Rightarrow$ The largest such interval is $(-1, 1)$. $\square$

Example 3. The second-order diff. eq.
$$y'' + 16y = 0$$

has a two-parameter family of solutions

$$y = C_1 \cos 4x + C_2 \sin 4x.$$

If we demand the initial condition

$$y\left(\frac{\pi}{2}\right) = -3, \quad y'\left(\frac{\pi}{2}\right) = 4,$$

then we have

$$C_1 \cos 2\pi + C_2 \sin 2\pi = -3$$

and

$$-4C_1 \sin 2\pi + 4C_2 \cos 2\pi = 4$$

Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, $C_1 = -3$ and $C_2 = 1$. Hence $y = -3 \cos 4x + \sin 4x$ is a sol of the IVP $\square$

Existence and Uniqueness

Two fundamental questions in the study of IVP are

- ① Does a solution of the problem exist?
- ② If a solution exists, is it unique?

Example 4. Each of the functions $y=0$ and $y = \frac{1}{16}x^4$ satisfies the diff. eq. $\frac{dy}{dx} = xy^{1/2}$ and the initial condition $y(0)=0$.

However, if we change the initial condition, say $y(0)=1$, then the above functions are not solutions of the IVP anymore.

Theorem. Existence of a Unique Solution

Let R be a rectangular region defined by $a \leq x \leq b$, $c \leq y \leq d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f / \partial y$ are continuous on R , then there exists an interval $I_0: (x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0 , that is a solution of the IVP:

$$\frac{dy}{dx} = f(x, y) \quad , \quad y(x_0) = y_0$$

Example 5. Consider the IVP in Example 4:

$$\frac{dy}{dx} = f(x, y) = x y^{1/2}, \quad y(0) = 0.$$

We have $\frac{\partial f}{\partial y} = \frac{x}{2 y^{1/2}}$ and $f(x, y) = x y^{1/2}$

are continuous on the upper half-plane defined by $y > 0$. Thus, the theorem tells us that through any point (x_0, y_0) , $y_0 > 0$, there is an interval centered at x_0 on which the given diff. eq. has a unique solution.

Thus, for example, even without solving it, we know that there exists an interval centered at 4 on which the IVP $dy/dx = x y^{1/2}$, $y(4) = 2$ has a unique solution. $\square$

In Example 1, Theorem guarantees that there are no other solutions of the IVPs $y' = y$, $y(0) = 3$ and $y' = y$, $y(1) = -2$ other than $y = 3e^x$ and $y = -2e^{x-1}$ respectively.

The interval of definition I is usually not as large as $\mathbb{R}$, and the interval I_0 of existence and uniqueness may not as large as I .

The number h in the defn of $I_0 = (x_0 - h, x_0 + h)$ could be very small, so we should think the solution $y(x)$ is "unique in a local sense".

